

Sequential Equilibrium and Perfect Bayesian Equilibrium

Sequential Equilibrium

- System of Beliefs

- At an information set I , a *belief* of the player who moves at I is a probability distribution over the nodes in information set I .

Given that this information set has been reached, what is the likelihood of being at each node in this information set? In games of imperfect information, the answer to such a question is essential for players' decision making.

- A *system of beliefs* specifies a belief for each information set in the game.

- Equilibrium in games of imperfect information comprises strategies and beliefs:

Beliefs are “consistent” with strategies, which are optimal given beliefs.

- Consistency of beliefs (Kreps and Wilson, 1982)

A system of beliefs, μ , is *consistent* with a profile of (behavioral) strategies, σ , if there exist a sequences of completely (strictly, totally) mixed (behavioral) strategy profiles $\{\sigma_k\}$ such that $\lim_{k \rightarrow \infty} (\sigma_k, \mu_k) = (\sigma, \mu)$, where μ_k is the system of beliefs derived from σ_k using Bayes' rule.

- *The idea is that beliefs at information sets that are reached with zero probability given σ (note that σ need not be completely mixed) must approximate beliefs that are derived from strategies that assign positive probability to every action.*
- *Consistency becomes trivial if σ itself is completely mixed.*

- Sequential rationality

- A strategy profile σ is *sequentially rational* at information set I given a system of beliefs μ if σ remains optimal for the player who moves at I if I is reached, given his subsequent beliefs (as embodied in μ) and others' subsequent strategies (as specified by σ). Formally, let i be the player who moves at I . Then σ is sequentially rational at I if $U_i(\sigma, \mu \mid I) \geq U_i((\sigma'_i, \sigma_{-i}), \mu \mid I)$ for every alternative strategy σ'_i , where $U_i(\sigma, \mu \mid I)$ is i 's expected utility at information set I given that the strategies and beliefs thereafter are specified by σ and μ .

(SR) σ is *sequentially rational* given μ if it is sequentially rational at *every* information set in the game.

- Sequential equilibrium
 (σ, μ) – an assessment – is a sequential equilibrium if σ is sequentially rational given μ and μ is consistent with σ .
- Sequential equilibrium and subgame perfect equilibrium
 If (σ, μ) is a sequential equilibrium, then σ is a subgame perfect equilibrium.
- Comments on sequential equilibrium: The trembles underlying the definition of consistency give all information sets positive probability so Bayes rule pin down beliefs everywhere. But
 - checking the consistency of beliefs (with a given strategy profile) is a tedious process;
 - one would like to know more about what consistency implies for behavior.

Perfect Bayesian Equilibrium

- The above definition of sequential equilibrium applies to games of imperfect information (and perfect recall). Definitions of Perfect Bayesian equilibrium (PBE) replace the consistency condition in the definition of sequential equilibrium with more intuitive restrictions on the way players update their beliefs as they play the game. One restriction common to all definitions of PBE is Bayesian updating on the equilibrium path. This restriction and sequential rationality gives rise to the notion of weak PBE (see, e.g., Mas-Colell et. al., 1995):
- (σ, μ) is a weak PBE if
 - (C1) The system of beliefs μ is derived from strategy profile σ through Bayes' rule whenever possible. To be more precise, if an information set I is reached with positive probability under σ (on the equilibrium path), then the belief at I is derived from σ using Bayes' rule.
 - (SR) σ is sequentially rational given μ .

This definition imposes Requirements 1, 2, and 3 In Gibbons (1992, p.p.177-178).

- It is easy to see that consistency implies (C1). Thus, every sequential equilibrium is also a weak PBE.
- Bayesian Nash equilibrium v.s. weak PBE:
 To see the difference between Bayesian Nash equilibrium and weak PBE, note that Bayesian Nash equilibrium can be defined as follows:
 A strategy profile σ is a Bayesian Nash equilibrium if there exists a system of beliefs μ such that

(C1) (see above)

- σ is sequentially rational at *all information sets that can be reached with positive probability* (on the equilibrium path) under σ .

- For the class of signalling games [see definitions in Gibbons (1992) or Tirole (1988)], weak PBE is equivalent to sequential equilibrium [see, e.g., Fudenberg and Tirole (1991)]. But in general, a weak PBE need not be a sequential equilibrium since consistency implies, in general, more than (C1). In fact, if (σ, μ) is a weak PBE, σ need not be a subgame perfect equilibrium. The reason is that the definition of weak PBE puts no restriction on players' beliefs at those information sets that are not reached with positive probability under σ . Gibbons' (1992) introduces additional condition (Requirement 4, p. 180) to define his version of PBE. This condition is as follows:

(C2) If an information set I is reached with zero probability under σ (off the equilibrium path), the belief at I is derived from σ using Bayes' rule, if possible.

- Gibbon's definition of PBE:
 (σ, μ) is PBE if (SR), (C1), and (C2).
- But, (C2) is evidently vague. From my reading of the literature, I have the following interpretation:

(C2') If an information set I is reached with zero probability under σ (off the equilibrium path), the belief at I is derived, using Bayes' rule, from the beliefs at the information sets that precede I and players' continuation strategies as specified by σ , if possible. [I think that this might be what Gibbons had in mind when he put forward (C2).]

- Thus our definition of PBE:
 (σ, μ) is PBE if (SR), (C1), and (C2').
- Such a definition of PBE ensure that if (σ, μ) is a PBE, σ is a subgame perfect equilibrium. But it does not address, for example, the issue of common beliefs – do 2 players with the same information have the same beliefs about a third player's type or unobserved moves? Therefore, we have to keep this in mind when applying the above definition of PBE particularly to games with more than 2 players. In Osborne and Rubinstein (1994) and Fudenberg and Tirole (1992), you can find definitions of PBE for games of incomplete information and observable actions (multistage games of incomplete information). Their definitions do address the issues like common beliefs. Fudenberg and Tirole (1992) provided conditions under which their definition of PBE is equivalent to sequential equilibrium. In general, sequential equilibrium is stronger (in that a sequential equilibrium gives rise to a PBE but not vice versa).

- For *signaling games* (a special type of games of incomplete information and observable actions), the aforementioned definitions of PBE are equivalent to the definition of weak PBE; moreover, they are all equivalent to sequential equilibrium.